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Invariant and partially-invariant solutions of the equations describing a non-stationary and isentropic flow for an ideal and compressible fluid in $(3 + 1)$ dimensions

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Abstract. This paper presents a new method of constructing, certain classes of solutions of a system of partial differential equations (PDEs) describing the non-stationary and isentropic flow for an ideal compressible fluid. A generalization of the symmetry reduction method to the case of partially-invariant solutions (PISs) has been formulated. We present a new algorithm for constructing PISs and discuss in detail the necessary conditions for the existence of non-reducible PISs. All these solutions have the defect structure $\delta = 1$ and are computed from four-dimensional symmetric subalgebras. These theoretical considerations are illustrated by several examples. Finally, some new classes of invariant solutions obtained by the symmetry reduction method are included. These solutions represent central, conical, rational, spherical, cylindrical and non-scattering double waves.

1. Introduction

This paper deals with the construction of several classes of exact solutions of the nonlinear system describing the isentropic flow of an ideal fluid. In order to construct these solutions, we apply the symmetry reduction method (SRM) to compute the invariant solutions and then we develop a method to construct partially-invariant solutions (PISs). The PISs have been introduced by Ovsiannikov [1] and can be seen as an extension of invariant solutions. So far, the method to obtain PISs, in the classical form, has been applied for systems with two independent and two dependent variables. We extend the applicability of this method to systems with an arbitrary number of unknown functions and independent variables. We develop a new algorithm for constructing these types of solutions. Furthermore, we are interested in so-called non-reducible PISs, i.e. solutions which are not invariant with respect to subgroups of the symmetry group of the governing system of equations.

We stress that the PISs are interesting for the following reasons: they can be constructed from an algorithm which is similar to the one used in the invariant case and is simple to use; PISs can solve larger classes of initial values problems compared to the invariant case; finally, we can obtain invariant solutions, with respect to low-dimensional groups, which would be very difficult to obtain directly by the SRM.

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Consider the classical ideal fluid dynamics equations in $(3+1)$ dimensions:

$$\begin{aligned} &\mathrm{D}\rho + \rho\,\mathrm{div}(\overline{u}) = 0 \\ &\mathrm{D}\overline{u} + \rho^{-1}\nabla p = 0 \\ &\mathrm{D}S = 0 \end{aligned} \tag{1.1}$$

where we use the following notation: ρ is the density of the fluid; p is the pressure of the fluid; $\overline{u} = (u, v, w)$ is the vector field of the fluid velocity; S is the entropy of the fluid; and D is the differential operator of the form $\mathrm{D} = \partial/\partial t + (\overline{u}\cdot\nabla)$. We will reduce the system of five quasilinear equations (1.1) in five unknown functions $(\rho, p, \overline{u})$ to the hyperbolic system describing an isentropic flow, according to [2].

The state equation of the media is given by

$$p = f(\rho, S). \tag{1.2}$$

If we eliminate the entropy from the first equation in (1.1) then

$$\mathrm{D}p = a^2\mathrm{D}\rho + f_S\mathrm{D}S$$

where $a^2 := f_\rho$. The third equation in (1.1) implies

$$\mathrm{D}p = a^2\mathrm{D}\rho$$

and using the first equation in (1.1) we get

$$\mathrm{D}p = -\rho a^2\,\mathrm{div}(\overline{u}).$$

The equations (1.1) become

$$\begin{aligned} &\mathrm{D}\rho + \rho\,\mathrm{div}(\overline{u}) = 0 \\ &\mathrm{D}\overline{u} + \rho^{-1}\nabla p = 0 \\ &\mathrm{D}p + \rho a^2\,\mathrm{div}(\overline{u}) = 0. \end{aligned} \tag{1.3}$$

The model of isentropic fluid requires that a^2 is a function of the density ρ only, i.e.

$$\nabla p = a^2(\rho)\nabla\rho$$

and

$$\frac{\mathrm{d}a}{a} = k^{-1}\frac{\mathrm{d}\rho}{\rho}$$

where $k = 2/(\gamma - 1)$ and γ is the adiabatic exponent. The velocity of sound is $a = (\gamma p/\rho)^{1/2}$. Therefore, the system of equations describing the non-stationary isentropic flow of a compressible ideal fluid takes the form

$$\begin{aligned} &\mathrm{D}a + k^{-1}a\,\mathrm{div}(\overline{u}) = 0 \\ &\mathrm{D}\overline{u} + ka\nabla a = 0. \end{aligned} \tag{1.4}$$

We denote the space of dependent variables by $U = \mathbb{R}^4$, where $(a, \overline{u}) \in U$, and the space of independent variables by $X = \mathbb{R}^4$, where $(x^\mu) = (t, x, y, z) \in X$, $\mu = 0, 1, 2, 3$. The system (1.4) represents a quasilinear hyperbolic system of four equations, written in the Cauchy–Kovalevski form. The largest Lie symmetry algebra of these equations has been investigated in [6]. It constitutes a Galilean similitude algebra generated by 12 operators. In the particular case when the adiabatic exponent $\gamma = \frac{5}{3}$, this algebra is generated by 13 infinitesimal operators: the 12 operators generating the Galilean similitude algebra and a projective transformation.

This paper is divided into five sections. In section 2, we present all the necessary background needed to understand the proposed algorithm for generating PISs. In section 3, these theoretical considerations are illustrated by several examples. In section 4, we present algebraical, spherical, cylindrical and conical invariant solutions. Section 5 contains some final remarks concerning the method of computation of PISs and their properties.

2. Construction of partially-invariant solutions

We start by giving all the theoretical notions needed to explain the algorithm of the computation of the PISs. This algorithm can be applied to any system of differential equations. Consider an nth order system of PDEs:

$$\Delta^i(x, u^{(n)}) = 0 \qquad i = 1, \ldots, m \tag{2.1}$$

with p independent variables ($x = (x^j) \in X$, $j = 1, \ldots, p$) and q dependent variables ($u = (u^l) \in U$, $l = 1, \ldots, q$). Suppose G_0 is a local symmetry group of the equations (2.1) which is fibre preserving (i.e. the transformation of the independent variables do not depend on the dependent on the dependent variables) and which acts regularly on the space $X \times U$. The orbits are r-dimensional and its Lie algebra L is generated by a set of infinitesimal generators, say $v_1, \ldots, v_s$ ($\dim(L) = s \geqslant r$). Suppose $u = f(x)$ is a solution of the equations (2.1), Γ_f is the graph of this solution ($\dim(\Gamma_f) = p$) and

$$G_0\Gamma_f = \{g \cdot (x, u) | (x, u) \in \Gamma_f, g \in G_0\} \tag{2.2}$$

where $g \cdot (x, u)$ is well defined. If Γ_f is relatively compact in the space $X \times U$, then $G_0\Gamma_f$ is a submanifold of $X \times U$. This set is called the orbit space of Γ_f (the union of the orbits of the Γ_f-elements) and it is the smallest G_0-invariant set which contains Γ_f. We then have the following properties: (i) if Γ_f is a G_0-invariant set (the solution $u = f(x)$ is G_0-invariant), then $G_0\Gamma_f = \Gamma_f$ and $\dim(G_0\Gamma_f) = p$; (ii) if Γ_f and all orbits of G_0 in $X \times U$ are transversal, then $\dim(G_0\Gamma_f) = p + \min(r, q)$ and these solutions are called generic. Consequently, we have the following inequalities:

$$p \leqslant \dim(G_0\Gamma_f) \leqslant p + \min(r, q). \tag{2.3}$$

When the inequalities are strict, $u = f(x)$ is a PIS. For a comprehensive review of the subject see, for example [3–5]. Ovsiannikov [1] has defined the defect structure of a solution to characterize such solutions. The parameter

$$\delta = \dim(G_0\Gamma_f) - \dim(\Gamma_f) = \dim(G_0\Gamma_f) - p \tag{2.4}$$

is called the defect structure of the solution $u = f(x)$ with respect to the group G_0. Therefore, a solution is (i) invariant if $\delta = 0$, (ii) partially invariant if $0 < \delta < \min(r, q)$ and (iii) generic if $\delta = \min(r, q)$. Then we can say that the PISs generalize invariant solutions.

The defect structure δ of a solution $u = f(x)$ can be calculated from the matrix of characteristics of the generators $v_1, \ldots, v_s$ (Ovsiannikov [1], p 277). Let

$$v_k = \sum_{i=1}^{p} \xi_k^i(x)\partial_{x^i} + \sum_{l=1}^{q} \phi_k^l(x, u)\partial_{u^l} \qquad k = 1, \ldots, s \tag{2.5}$$

be these infinitesimal generators. The matrix of characteristics (of dimension $q \times s$) with respect to these generators is

$$Q(x, u^{(1)}) = (b_\beta^\alpha(x, u^{(1)})) \qquad \alpha = 1, \ldots, q \qquad \beta = 1, \ldots, s \tag{2.6}$$

where $b_\beta^\alpha(x, u^{(1)}) = \phi_\beta^\alpha(x, u) - \sum_{i=1}^{p} \xi_\beta^i(x)\partial u^\alpha/\partial x^i$. Then $u = f(x)$ is a PIS of equations (2.1), with defect structure with respect to G_0 equal to δ, if only if

$$\operatorname{rank}(Q(x, u^{(1)})_{|u=f(x)} = \delta. \tag{2.7}$$

From this result, we can formulate a first method of calculation of the PISs. We construct an augmented system which is composed of (a) the basis equations (2.1) and (b) the first-order system, obtained by demanding that all determinants of the minors of order $(\delta+1)\times(\delta+1)$ of the matrix of characteristics (2.6) are equal to zero. A solution of the augmented system is thus a PIS of the system (2.1), with defect structure δ with respect to the group G_0. For example, when $\delta = 1$, all the determinants of 2×2 minors of the matrix (2.6) vanish. This method, however, has the disadvantage that it leads to many calculations with nonlinear equations. Furthermore, we cannot distinguish the groups which will give genuine PISs, i.e. solutions which are not invariant. In order to overcome these difficulties (to a certain extent), we have developed an alternative algorithm of calculation.

To apply this algorithm, we need to determine the largest symmetry group of the system (2.1). This group is supposed to be fibre preserving and we denote it by $\bar{H}$. We refer to [7] for the calculations of this group. Then, we classify the subgroups of $\bar{H}$ into conjugacy classes. The method of classification which has been employed is presented in [8] and [9]. The subgroups of the group $\bar{H}$, which have been chosen for the calculations of the PISs, are as follows. Let $H_i \subset \bar{H}$ be a representative of a conjugacy class. The subgroup H_i is fibre preserving and induces an action on the space X. If the dimension of the H_i orbits on the space $X\times U$ is s, then this subgroup will generate a PIS with defect structure δ if the dimension of the orbits on the space X is $s-\delta$. Then the subgroup has $p+\delta-s$ invariants which depend only on x and $q-\delta$ invariants which in turn depend on x and u. We select the subgroups H_i which have this characteristic. We note that a subgroup H_k, which is fibre preserving, generates H_k-invariant solutions if the dimension of the orbits of H_k in the space $X\times U$ are equal to the dimension of the orbits on the space X.

2.1. The algorithm for the calculation of PISs

Now, we give the description of our algorithm of the calculation. We suppose that $H_i \subset \bar{H}$ is a subgroup with s-dimensional orbits in the space $X\times U$, and $(s-\delta)$-dimensional orbits in the space X. The solutions which are obtained from this group will be PISs with the defect structure equal to δ. The procedure involves several steps.

(i) Construct a complete set of functionally independent invariants for H_i. If $\{v_1,\dots,v_s\}$ is a basis of infinitesimal generators of the Lie algebra $L_i = \exp(H_i)$, then I is an invariant of H_i if and only if $v_j(I) = 0$, $j = 1,\dots,s$. We obtain a set of functionally independent invariants of the form

$$\{\eta^k(x), I^j(x,u)\} \tag{2.8}$$

where $k = 1,\dots,p+\delta-s$ and $j = 1,\dots,q-\delta$. Then we have

$$\operatorname{rank}\left(\frac{\partial(I^1(x,u),\dots,I^{q-\delta}(x,u))}{\partial(u^1,\dots,u^q)}\right) = q-\delta. \tag{2.9}$$

(ii) Express the $(p+\delta)$-dimensional manifold $H_i\Gamma_f$ in terms of the invariants (2.8), where Γ_f is the graph of a function $u = f(x)$. $H_i\Gamma_f$ is the smallest invariant manifold containing Γ_f, with respect to the action of the group H_i. These equations take the form

$$I^j(x,u) = f^j(\eta^k(x)) \tag{2.10}$$

where $j = 1,\dots,q-\delta$ and $k = 1,\dots,p+\delta-s$. The functions f^j are arbitrary.

(iii) From (2.9) and the implicit functions theorem, we have for $q-\delta$ coordinates in U

$$u^{i_a} = \Phi^{i_a}(f^j(\eta^k(x), x, u^{j_l})) \tag{2.11}$$

where $a = 1, \ldots, q - \delta$, $l = 1, \ldots, \delta$, $j = 1, \ldots, q - \delta$ and $k = 1, \ldots, p + \delta - s$. There are no constraints on the δ remaining coordinates in U and we put

$$u^{jl} = \varphi^{jl}(x) \tag{2.12}$$

where $l = 1, \ldots, \delta$ and the functions φ^{jl} are arbitrary.

(iv) From equations (2.11) and (2.12), we calculate the derivatives of the $u^l (l = 1, \ldots, q)$ and we substitute these expressions into equations (2.1).

(v) Suppose that the rank of the system obtained from (iv) is r_δ relative to the derivatives of the δ functions given in (2.12). Thus we need to solve this system of equations for r_δ of these derivatives. From expressions (iv), we calculate the compatibility conditions for this system of equations. From these constraints, we obtain a PDE system, denoted by Δ^j/H^i, expressed in terms of only the q–δ functions f^j of (2.10) and the invariants $\eta^k(x)$, and a system of PDEs denoted by Δ^1, for the functions φ^{jl} of (2.12).

(vi) Solve the system Δ^j/H^i.

(vii) For each solution of Δ^j/H^i, solve the system Δ^1.

(viii) Substitute the solutions calculated in (vi) and (vii) into the equations given in (2.11) and (2.12) to obtain a PIS with defect structure δ for the system (2.1).

In the case of invariant solutions, there is a one-to-one correspondance between the solutions of the reduced system and the invariant solutions of the basic system. For PISs, we do not have such a correspondance. Indeed, for every solution of the system Δ^j/H^i, we have to solve the system Δ^1 to obtain a PIS for the system Δ^j. Thus, to a single solution of the system Δ^j/H^i, there may correspond a family of solutions of the system Δ^1. We will encounter such a situation when there are solutions of Δ^j/H^i for which no solutions of Δ^1 are compatible with Δ^j.

Once these calculations are completed, we could check whether the solutions so obtained are invariant with respect to some subgroups of the symmetry group. The calculations needed for the symmetry group of high dimension are generally hard to perform, but we may restrict our considerations to the subgroups of the group H_i from which we have derived our original solutions. Ovsiannikov [1] has formulated the concept of reducibility for such a problem.

A PIS $u = f(x)$, with respect to a group H_i, is called reducible if (i) there exists a subgroup $H_a \subset H_i$ for which $u = f(x)$ is H_a-invariant and (ii) $\dim(H_a \Gamma_f) = s^1$ where $s^1 \geqslant s - \delta$. We are interested in non-reducible PISs, since reducible solutions can be calculated from reduced systems involving $p - s^1$ independent variables where $p - s^1 \leqslant p + \delta - s$. Therefore, these reduced systems are easier to solve than the systems Δ^j/H^i and Δ^1 which we have to solve to obtain PISs.

Now we present the results that we have obtained for the system (1.4), with examples illustrating the calculations of PISs.

3. Examples of applications

We want to calculate PISs of the system (1.4) from subgroups H of the largest symmetry group of this system which have the following properties: (i) the defect structure of the solutions is $\delta = 1$; (ii) Δ^j/H is a system of ordinary differential equations (ODEs). Therefore, we have to consider four-dimensional subgroups. Furthermore, we want to construct certain classes of non-reducible solutions, i.e. solutions which are not invariant with respect to the three-dimensional subgroups of H.

The Lie symmetry algebra of the system (1.4) is generated by the following infinitesimal generators:

(i) when $\gamma \neq 5/3$ ($k \neq 3$),

$$P_\mu = \partial_{x^\mu} \qquad J_k = \varepsilon_{kij}(x^i \partial_{x^j} + u^i \partial_{u^j}) \qquad K_i = x^0 \partial_{x^i} + \partial_{u^i}$$
$$F = x^\mu \partial_{x^\mu} \qquad G = -x^0 \partial_{x^0} + u^\mu \partial_{u^\mu} \tag{3.1}$$

(ii) when $\gamma = 5/3$ ($k = 3$), the 12 infinitesimal generators which appear in (3.1) and the projective transformation

$$C = x^0(x^\mu \partial_{x^\mu} - u^0 \partial_{u^0}) + (x^i - x^0 u^i)\partial_{u^i} \tag{3.2}$$

where $i, j, k = 1, 2, 3$ and $\mu = 0, 1, 2, 3$. We note that this algebra is fibre preserving. In [6], we present the classification into conjugacy classes of the subalgebras of dimension from one to four inclusively, of the algebra generated by the operators (3.1) and (3.2).

We start by giving examples illustrating the calculations involved in the algorithm which help us to understand the results presented in our examples. Furthermore, we use the Lie algebras of the subgroups, rather than the subgroups themselves, which are more convenient for performing our calculations.

Example 1. Consider the Lie algebra $\{K_1, K_2, K_3, P_3\}$. The set of functionally independent invariants is

$$\{x - ut, y - vt, a, t\}. \tag{3.3}$$

Then we have

$$\operatorname{rank}\left(\frac{\partial(x - ut, y - vt, a)}{\partial(u, v, w, a)}\right) = 3.$$

The PIS will have a defect structure $\delta = 1$. The equations, giving the orbits of the solutions, are of the form

$$x - ut = F(t) \qquad y - vt = G(t) \qquad a = A(t) \tag{3.4}$$

corresponding to equations (2.10) and (2.11). Then the expression for the solutions is

$$u = \frac{x - F}{t} \qquad v = \frac{y - G}{t} \qquad w = w(t, x, y, z) \qquad a = A(t). \tag{3.5}$$

Note that no constraints have been imposed on the function w. We calculate the derivatives of functions u, v, w and a from equations (3.5):

$$\begin{aligned} &u_t = \frac{F - x}{t^2} - \frac{F'}{t} \qquad u_x = \frac{1}{t} \qquad u_y = u_z = 0 \\ &v_t = \frac{G - y}{t^2} - \frac{G'}{t} \qquad v_x = v_z = 0 \qquad v_y = \frac{1}{t} \\ &a_t = A' \qquad a_x = a_y = a_z = 0. \end{aligned} \tag{3.6}$$

Substituting these expressions into system (1.4) gives

$$A' + \frac{A}{k}\left(\frac{2}{t} + w_z\right) = 0 \tag{3.7}$$
$$F' = 0 \qquad G' = 0 \tag{3.8}$$
$$w_t + \frac{(x - F)}{t} w_x + \frac{(y - G)}{t} w_y + w w_z = 0. \tag{3.9}$$

From (3.8), we obtain $F = C_1$, $G = C_2$, where C_1, C_2 are arbitrary constants. From (3.7), we get

$$w = -\left(\frac{kA'}{A} + \frac{2}{t}\right) z + \Psi(t, x, y) \tag{3.10}$$

where we denote $\Phi(t) = kA'/A + 2/t$. Then

$$w_t = -\Phi z + \Psi_t \qquad w_x = \Psi_x \qquad w_y = \Psi_y \qquad w_z = -\Phi. \tag{3.11}$$

We substitute these derivatives into (3.9) and obtain

$$\Psi_t + \frac{(x - C_1)}{t}\Psi_x + \frac{(y - C_2)}{t}\Psi_y - \Phi\Psi + z(\Phi^2 - \Phi') = 0. \tag{3.12}$$

Therefore, we must have

$$\Psi_t + \frac{(x - C_1)}{t}\Psi_x + \frac{(y - C_2)}{t}\Psi_y - \Phi\Psi = 0 \tag{3.13}$$

$$\Phi^2 - \Phi' = 0. \tag{3.14}$$

The system Δ^i/H consists of equations (3.8) and (3.14), and the system Δ^1 is described by equation (3.13). From (3.14), we obtain

$$\Phi = \frac{-1}{t + C_3} = \frac{kA'}{A} + \frac{2}{t}. \tag{3.15}$$

Therefore,

$$A = C_4(t^{-2}(t + C_3))^{-1/k} \tag{3.16}$$

where C_3 and C_4 are arbitrary constants. We solve equations (3.13) by the method of characteristics and we obtain

$$\Psi = (t + C_3)^{-1}\lambda(\xi_1, \xi_2) \tag{3.17}$$

where: $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary function and $\xi_1 = t(x - C_1)^{-1}$, $\xi_2 = t(y - C_2)^{-1}$. Then we obtain the solution

$$u = \frac{x - C_1}{t} \qquad v = \frac{y - C_2}{t} \qquad w = \frac{z + \lambda(\xi_1, \xi_2)}{t + C_3} \qquad a = C_4\left(\frac{t + C_3}{t^2}\right)^{1/k}. \tag{3.18}$$

Next, we determine the solutions which are invariant with respect to one parameter subgroups of the group discussed in the present example. These subgroups have their Lie algebras generated by an infinitesimal generator of the form

$$\bar{v} = a_1K_1 + a_2K_2 + a_3K_3 + a_4P_3 \qquad (a_i \in \mathbb{R}).$$

We have

$$\bar{v}\left(u - \frac{x - C_1}{t}\right) = \bar{v}\left(v - \frac{y - C_2}{t}\right) = \bar{v}\left(a - C_4\left(\frac{t + C_3}{t^2}\right)^{-1/k}\right) = 0. \tag{3.19}$$

Thus we must put

$$\bar{v}\left[w - \left(\frac{z + \lambda(\xi_1, \xi_2)}{t + C_3}\right)\right] = 0$$

which gives the equation

$$a_1\xi_1^2\lambda_{\xi_1} + a_2\xi_2^2\lambda_{\xi_2} = a_4 - a_3C_3. \tag{3.20}$$

Then

(i) if $a_1a_2 \neq 0$,

$$\lambda = (a_4 - a_3C_3)\left(\frac{-1}{a_1\xi_1} + \varphi(X)\right) \tag{3.21}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an arbitrary function and $X = 1/a_2\xi_2 - 1/a_1\xi_1$;

(ii) if $a_1 = 0$ and $a_2 \neq 0$,

$$\lambda = \frac{a_3C_3 - a_4}{a_2}\xi_2^{-1} + \varphi(\xi_1) \tag{3.22}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an arbitrary function;

(iii) if $a_1 \neq 0$ and $a_2 = 0$,

$$\lambda = \frac{a_3C_3 - a_4}{a_1}\xi_1^{-1} + \varphi(\xi_2) \tag{3.23}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an arbitrary function.

We note that this solution is invariant relative to the algebra $\{K_3 + C_3P_3\}$. A solution which is not of the form given in (3.21), (3.22) or (3.23) is not invariant relative to any algebra which has non-zero component onto the space generated by K_1 and K_2. It is thus a non-reducible PIS.

From this first example, we observe that non-reducible solutions can be constructed due to the presence of an arbitrary function λ in w. Indeed, with four-dimensional algebras and defect structure $\delta = 1$, reducible solutions are invariant relative to three-dimensional subalgebras. Then the corresponding reduced systems are ODEs and the invariant solutions contain only arbitrary constants. Then, in this case, the existence of an arbitrary function in a PIS assures the existence of non-reducible solutions which are obtained from equivalent constraints given by equation (3.20).

In the next example, we perform a change of variables to facilitate the computation of the PIS. Furthermore, the functions on which no constraints are imposed are not included among the variables u, v, w or a, but among the newly-defined variables.

Example 2. Consider the algebra $\{J_3, P_1, P_2, P_3\}$. We obtain the set of functionally independent invariants

$$\{(u^2 + v^2)^{1/2}, w, a, t\}. \tag{3.24}$$

We make the change of variables

$$u = r\cos\theta \qquad v = r\sin\theta \tag{3.25}$$

and the new invariants are

$$r,\ w,\ a,\ t. \tag{3.26}$$

The new invariants simplify the calculations, compared to those given in (3.24). We have

$$\operatorname{rank}\left(\frac{\partial(r, w, a)}{\partial(r, w, a, \theta)}\right) = 3. \tag{3.27}$$

Thus the PIS which we will calculate will have defect structure $\delta = 1$. The equations giving the orbits of this solution are

$$r = R(t) \qquad w = H(t) \qquad a = A(t). \tag{3.28}$$

Then, the expressions for this solution are given in (3.28) and we obtain u and v from (3.25), where $\theta = \theta(t, x, y, z)$. We calculate the derivatives of the functions u, v, w and a and we obtain the following system of equations:

$$\begin{aligned} &A' + \frac{RA}{k}[\cos(\theta)\theta_y - \sin(\theta)\theta_x] = 0 \qquad R' = 0 \\ &\theta_t + R\cos(\theta)\theta_x + R\sin(\theta)\theta_y + H\theta_z = 0 \qquad H' = 0. \end{aligned} \tag{3.29}$$

Then $R = C_1$ and $H = C_2$, where C_1, C_2 are arbitrary real constants. We solve the system (3.29) for θ_x and θ_y and we obtain

$$\begin{aligned} \theta_x &= C_1^{-1}[-\cos(\theta)(\theta_t + C_2\theta_z) + k\sin(\theta)\rho'] \\ \theta_y &= -C_1^{-1}[\sin(\theta)(\theta_t + C_2\theta_z) + k\cos(\theta)\rho'] \end{aligned} \tag{3.30}$$

where $\rho(t) = \ln(A)$. The compatibility conditions, obtained from (3.30), give the equation

$$\rho'' - k(\rho')^2 = 0. \tag{3.31}$$

The solution of equation (3.31) is

$$\rho = \ln(C_4 t + C_5)^{-1/k}$$

and so

$$A = (C_4 t + C_5)^{-1/k}.$$

Thus we obtain

$$\theta = \varphi(\xi_1, \xi_2, \xi_3) \tag{3.32}$$

with φ being implicitly defined by the equation

$$\varphi = \cos^{-1}\left[\frac{C_4}{C_1C_5}\xi_1 + F\left(\frac{-C_1C_5}{C_4}\sin(\varphi) + \xi_2, \xi_3\right)\right] \tag{3.33}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary function, $\xi_1 = x - C_1 t\cos(\theta)$, $\xi_2 = y - C_1 t\sin(\theta)$ and $\xi_3 = z - C_2 t$. Hence the solution is

$$u = C_1\cos(\theta) \qquad v = C_1\sin(\theta) \qquad w = C_2 \qquad a = (C_4 t + C_5)^{-1/k} \tag{3.34}$$

where θ is defined from (3.32) and (3.33). We have obtained non-reducible solutions because (3.34) contains an arbitrary function. Such functions are determined exactly in the same manner as in example 1.

In table 1, we present several examples of PISs which we have obtained from four-dimensional subalgebras of the symmetry algebra of system (1.4). We find in this table the following information: (a) the first column gives the algebras from which we compute the solutions; (b) in the second column we give the invariants of the algebras; (c) in the third column we give the expressions of the dependent variables, corresponding to equations (2.11) and (2.12); (d) in the fourth column we give the solutions; and (e) in the last column we specify parameters which appear in the expressions of the solutions. Note that the solution, corresponding to the algebra $\{P_2 + K_3, K_2, P_3, P_1\}$, is not invariant with respect to any three-dimensional subalgebras of the symmetry algebra of equations (1.4).

Table 1. Non-reducible partially-invariant solutions of the system of equations for isentropic flows in $(3+1)$ dimensions ($\lambda : \mathbb{R}^2 \to \mathbb{R}$ represents an arbitrary function and the C_i's represent arbitrary constants).

Algebras	Invariants	Dependent variables	Solutions	Remarks
$\{K_1, K_2, P_1, P_3\}$	$\{y - vt, w, a, t\}$	$u = u(t, x, y, z)$	$u = \frac{x+\lambda(\xi_1,\xi_2)}{t-C_1}$	$\xi_1 = \frac{t}{y-C_2}$
		$v = \frac{y-G(t)}{t}$	$v = \frac{y-C_2}{t}$	$\xi_2 = z - C_3 t$
		$w = H(t)$	$w = C_3$	
		$a = A(t)$	$a = C_4(t(t - C_1))^{-1/k}$	
$\{K_1, K_2, P_1, P_3 + \alpha P_2\}$ $\alpha > 0$	$\{y - \alpha z - vt, w, a, t\}$	$u = u(t, x, y, z)$	$u = \frac{x+\lambda(\xi_1,\xi_2)}{t-C_1}$	$\xi_1 = C_3 t - z$
		$v = \frac{y-\alpha z-G(t)}{t}$	$v = \frac{y+\alpha C_3 t-\alpha z-C_2}{t}$	$\xi_2 = \frac{t}{y+\alpha C_3 t-\alpha z-C_2}$
		$w = H(t)$	$w = C_3$	
		$a = A(t)$	$a = C_4(t(t - C_1))^{-1/k}$	
$\{K_1 + P_3, K_2, P_1, P_2\}$	$\{z - u, w, a, t\}$	$u = z - F(t)$	$u = z - C_3 t - C_1$	$\xi_1 = z - C_3 t$
		$v = v(t, x, y, z)$	$v = \frac{y+\lambda(\xi_1,\xi_2)}{t+C_2}$	$\xi_2 = C_3 x - (z - C_3 t - C_1)z$
		$w = H(t)$	$w = C_3$	
		$a = A(t)$	$a = C_4(t + C_2)^{-1/k}$	
$\{P_1, P_2 P_3, K_3\}$	$\{u, v, a, t\}$	$u = F(t)$	$u = C_1$	$\xi_1 = x - C_1 t$
		$v = -G(t)$	$v = C_2$	$\xi_2 = y - C_2 t$
		$w = w(t, x, y, z)$	$w = \frac{z+\lambda(\xi_1,\xi_2)}{t+C_3}$	
		$a = A(t)$	$a = C_4(t + C_3)^{-1/k}$	
$\{P_2 + K_3, K_2, P_3, P_1\}$	$\{u, y - vt - w, a, t\}$	$u = F(t)$	$u = C_2$	$g : \mathbb{R} \to \mathbb{R}$, arbitrary
		$v = v(t, x, y, z)$	$v = \frac{(p-qt)y+qz-qC_3 t+g(\xi)}{1+pt-qt^2}$	$\xi = x - C_2 t$
		$w = y - vt + H(t)$	$w = \frac{y-qtz+C_3(1+pt)-tg(\xi)}{1+pt-qt^2}$	$p, q \in \mathbb{R}$
		$a = A(t)$	$a = \frac{C_1}{(1+pt-qt^2)^{1/2}}$	

4. Invariant solutions

The graph of a G_0-invariant solution is an invariant set, with respect to the group G_0, unlike a PIS. The group G_0 must also satisfy the following two conditions: (i) the dimension of the orbits in the spaces X and $X \times U$ is the same; and (ii) $\dim(G_0) < p$. The algorithm to construct these solutions can be obtained from the one presented in section 2 by setting $\delta = 0$. We obtain a reduced system, which is expressed only in terms of the invariants and their derivatives with respect to the symmetry variable, from which we obtain the invariant solutions. Now, we present several examples which illustrate the calculation of invariant solutions. In [10], an optimal system of invariant solutions has been obtained for the system (1.4), with respect to the three-dimensional subgroups of the symmetry group of this system.

Example 3. Cylindrical solutions. We consider the algebra generated by

$$J_3 = x\partial_y - y\partial_x + u\partial_v - v\partial_u \qquad P_0 = \partial_t \qquad P_3 = \partial_z. \tag{4.1}$$

The global invariants are

$$\begin{aligned} &s = (x^2 + y^2)^{1/2} \qquad \Phi^1 = (u^2 + v^2)^{1/2}, w, a \\ &\Phi^2 = \sin^{-1}\left(\frac{x}{(x^2 + y^2)^{1/2}}\right) - \sin^{-1}\left(\frac{u}{(u^2 + v^2)^{1/2}}\right) \end{aligned} \tag{4.2}$$

where $x^2 + y^2 \neq 0$, $u^2 + v^2 \neq 0$ and s is the symmetry variable. As our symmetry variable is s, the invariant solution will be of the cylindrical type. We form the expressions

$$\begin{aligned} &(u^2 + v^2)^{1/2} = F(s) \qquad \sin^{-1}\left(\frac{x}{s}\right) - \sin^{-1}\left(\frac{u}{F}\right) = G(s) \\ &w = H(s) \qquad a = A(s) \end{aligned} \tag{4.3}$$

which give us an expression for an invariant function in the space of the invariants. Then we have

$$\begin{aligned} &u = \frac{F}{s}(x\cos(G) - y\sin(G)) \qquad v = \frac{F}{s}(y\cos(G) + x\sin(G)) \\ &w = H \qquad a = A. \end{aligned} \tag{4.4}$$

From this, we calculate the partial derivatives of the dependent variables and we obtain

$$\begin{aligned} &u_x = \frac{x}{s^2}(x\cos(G) - y\sin(G))F' + \frac{F}{s^2}(y\cos(G) + x\sin(G))\left(\frac{y}{s} - xG'\right) \\ &u_y = \frac{y}{s^2}(x\cos(G) - y\sin(G))F' - \frac{F}{s^2}(y\cos(G) + x\sin(G))\left(\frac{x}{s} + yG'\right) \\ &u_t = u_z = 0 \\ &v_x = \frac{x}{s^2}(y\cos(G) + x\sin(G))F' - \frac{F}{s^2}(x\cos(G) - y\sin(G))\left(\frac{y}{s} - xG'\right) \\ &v_y = \frac{y}{s^2}(y\cos(G) + x\sin(G))F' + \frac{F}{s^2}(x\cos(G) - y\sin(G))\left(\frac{x}{s} + yG'\right) \\ &v_t = v_z = 0 \\ &w_t = w_z = 0 \qquad w_x = \frac{x}{s}H' \qquad w_y = \frac{y}{s}H' \\ &a_t = a_z = 0 \qquad a_x = \frac{x}{s}A' \qquad a_y = \frac{y}{s}A'. \end{aligned} \tag{4.5}$$

We substitute expressions (4.5) into (1.4), and we obtain the equations

$$F\cos(G)A' + \frac{A}{k}\left(\cos(G)F' + \left(\frac{\cos(G)}{s} - \sin(G)G'\right)F\right) = 0 \tag{4.6}$$

$$\cos(G)(x\cos(G) - y\sin(G))FF' - \cos(G)(y\cos(G) + x\sin(G))F^2G' \\ -\frac{\sin(G)}{s}(x\sin(G) + y\cos(G))F^2 + kxAA' = 0 \tag{4.7}$$

$$\cos(G)(y\cos(G) + x\sin(G))FF' + \cos(G)(x\cos(G) - y\sin(G))F^2G' \\ +\frac{\sin(G)}{s}(x\cos(G) - y\sin(G))F^2 + kyAA' = 0 \tag{4.8}$$

$$s\cos(G)H' = 0. \tag{4.9}$$

These four equations do not form the reduced system because equations (4.7) and (4.8) are not expressed only in terms of the invariants. However, the following combinations of the equations, with variable coefficients,

$$x(4.8) - y(4.7) \qquad \text{and} \qquad x(4.7) + y(4.8) \tag{4.10}$$

eliminate all non-invariant terms and we obtain the equations

$$\sin(G)FF' + \cos(G)F^2G' + s^{-1}\sin(G)F^2 = 0 \tag{4.11}$$

$$s^2\cos^2(G)FF' - s^2\cos(G)\sin(G)F^2G' - s\sin^2(G)F^2 + ks^2AA' = 0. \tag{4.12}$$

The reduced system is thus formed by equations (4.6), (4.9), (4.11) and (4.12). Our method has led us to two types of solution: F and H are arbitrary functions of the variable s,

$$G = \frac{(2k+1)}{2}\pi (k \in N) \qquad \text{and} \qquad A = \pm\left(\frac{2}{k}\int\frac{F^2}{s}\,\mathrm{d}s\right)^{1/2} \tag{4.13}$$

and

$$F = C_1(s\sin(G))^{-1} \qquad H = C_4 \qquad A = C_2\left[\frac{\tan(G/2)(\tan(G/2)+1)}{1-\tan(G/2)}\right]^{1/k} \tag{4.14}$$

where G is defined implicitly by the equation

$$s^2\sin^2(G)\left[-2\frac{C_2^2}{C_1^2}\int\left(\frac{\tan(G/2)(1+\tan(G/2))^{2/k}}{1-\tan(G/2)}\right)\frac{1}{\sin(G)\cdot\cos(G)}\,\mathrm{d}G + C_3\right] - 1 = 0$$

where $C_1 \neq 0$, $(2k+1)\pi < G < ((4k+3)/2)\pi$ or $2k\pi < G < ((4k+1)/2)\pi$ ($k \in \mathbb{Z}$), which guarantee the existence of real non-singular solutions. Since we have explicit expressions for the functions F, G, H and A, we substitute them into equations (4.4) and we find an invariant solution of the system (1.4) with respect to the algebra considered.

In the next example, we show that an algebra which should give only PISs can give invariant solutions under some conditions on the functions u, v, w and a. This solution is of spherical type.

Example 4. Spherical solutions. We consider the algebra $\{J_1, J_2, J_3\}$. The global invariants of this algebra are:

$$t \qquad s = (x^2+y^2+z^2)^{1/2} \qquad \Phi^1 = (u^2+v^2+w^2)^{1/2}$$
$$\Phi^2 = xu + yv + zw \qquad \Phi^3 = a.$$

We have

$$\operatorname{rank}\left(\frac{\partial(\Phi^1,\Phi^2,\Phi^3)}{\partial(u,v,w,a)}\right) = 3.$$

Therefore, we cannot obtain invariant solutions from these invariants by the SRM. The equations giving the orbit of the graph are

$$
\begin{aligned}
&u = u(t,x,y,z) \qquad v = \frac{g(s,t) - xu - zw}{y} \\
&w = \frac{z(xs - g(u,t)) \pm y[(f^2(s,t) - u^2)(y^2+z^2) - (xu - g(s,t))^2]^{1/2}}{y^2+z^2} \\
&a = A(s,t)
\end{aligned}
\tag{4.15}
$$

where $\Phi^1 = f(s,t)$ and $\Phi^2 = g(s,t)$ are arbitrary functions. Substituting equation (4.15) into (1.4) gives us the reduced system

$$
\begin{aligned}
&A_t + \frac{gA_s}{s} + \frac{A}{k}\Bigg[u_x + \left(\frac{xyz(xu-g) + zyu(y^2+z^2) - \varphi xy^2}{y(y^2+z^2)\varphi}\right) \\
&\qquad \times u_y - \left(\frac{zx\varphi + (y^2+z^2)yu + xy(xu-g)}{(y^2+z^2)\varphi}\right)u_z \\
&\qquad + \frac{g_s}{s} - \frac{z(z^2+y^2-1)}{y(y^2+z^2)}\varphi + \frac{z^2(z^2+y^2-1)}{y^2(y^2+z^2)^2}(xu-g)\Bigg] = 0 \\
&u_t + uu_x - \left(\frac{y(xu-g)+z\varphi}{y^2+z^2}\right)u_y + \left(\frac{y\varphi - z(xu-g)}{y^2+z^2}\right)u_z + \frac{kxAA_s}{s} = 0 \\
&g_t + \frac{gg_s}{s} + ksAA_s - f^2 = 0 \\
&g_t + \frac{gg_s}{s} + k\left(\frac{x^2+y^2}{s} + z\right)AA_s - f^2 = 0
\end{aligned}
\tag{4.16}
$$

where $\varphi = [(f^2 - u^2)(y^2+z^2) - (xu-g)^2]^{1/2}$. To compute the solutions of (4.16), we have to satisfy the compatibility conditions (see the algorithm presented in section 2.1). Thus the solution will be a non-reducible PIS, because this algebra does not contain any two-dimensional subalgebras. We will not continue with these calculations here, but we will show that we can obtain solutions with spherical symmetry by considering functions with a special form.

In order to do this, we write the functions u, v, w and a in the following form:

$$
u^\mu = x^\mu f(s,t) \qquad a = A(s,t) \qquad \mu = 1,2,3 \tag{4.17}
$$

where f and A will be determined from the condition that the functions defined in (4.17) are solutions of (1.4). Since

$$
J_i(u^\mu - x^\mu f(s,t)) = 0 \qquad J_i(a - A(s,t)) = 0 \qquad \mu, i = 1,2,3
$$

the functions in (4.17) are invariant with respect to the algebra $\{J_1, J_2, J_3\}$. This case corresponds to a non-stationary flow with spherical symmetry, which is irrotational. System (1.4) is thus reduced to the equations

$$
A_t + sfA_s + k^{-1}A(3f + sf_s) = 0 \qquad f_t + sff_s + ks^{-1}AA_s + f^2 = 0. \tag{4.18}
$$

We look for solutions of (4.18) by the method of separation of variables. Hence, we put $A = \alpha_1(t)h_1(s)$, $f = \alpha_2(t)h_2(s)$. We obtain

$$
\alpha_1 = \epsilon k^{-1/2}t^{-1} \qquad \alpha_2 = \lambda_1^{-1}t^{-1} \qquad (\lambda_1 \in \mathbb{R} \text{ and } \lambda_1 \neq 0,\ t > 0,\ \epsilon = \pm 1) \tag{4.19}
$$

and the non-autonomous ODE system

$$
\frac{\mathrm{d}h_1}{\mathrm{d}s} = \frac{sh_1h_2((k-1)\lambda_1 - 2h_2)}{(ks^2h_2^2 - \lambda_1^2h_1^2)} \qquad \frac{\mathrm{d}h_2}{\mathrm{d}s} = \frac{k\lambda_1 s^2h_2^2 + 3\lambda_1^2h_1^2h_2 - k\lambda_1^3h_1^2 - ks^2h_2^2}{s(ks^2h_2^2 - \lambda_1^2h_1^2)}
$$

whose solutions exist if $h_1^2 \neq \lambda_1^{-2} k s^2 h_2^2$. For the stationary case (i.e. if f and A do not depend on t in expressions (4.17)) we obtain the solution

$$u^\mu = C_1 x^\mu s^{-(k+3)/k+1} \qquad a = \epsilon(-k)^{1/2} C_1^{-1/k} s^{-2/k+1} \qquad \mu = 1, 2, 3$$

where $\epsilon = \pm 1$, $C_1 > 0$ and $x^2 + y^2 + z^2 > 0$. These conditions guarantee the existence of real and non-singular solutions.

Example 5. Conical solutions. We consider the algebra $\{J_1, C, K_1\}$. The corresponding invariants are the symmetry variable $s = t^{-1}(y^2 + z^2)^{1/2}$, $F(s) = x - ut$,

$$G(s) = \sin^{-1}\left(\frac{y}{(y^2+z^2)^{1/2}}\right) - \sin^{-1}\left(\frac{y - vt}{((y-vt)^2 + (z-wt)^2)^{1/2}}\right)$$
$$H(s) = ((y-vt)^2 + (z-wt)^2)^{1/2}$$

and $A(s) = at$. Then, the unknown functions take the form

$$\begin{aligned} u &= \frac{x - F}{t} \qquad v = \frac{y}{t} - \frac{H}{st^2}(y\cos(G) - z\sin(G)) \\ w &= \frac{z}{t} - \frac{H}{st^2}(z\cos(G) + y\sin(G)) \qquad a = \frac{A}{t}. \end{aligned} \tag{4.20}$$

We obtain the reduced system

$$\begin{aligned} &HA' + \frac{A}{3}\left(H' - H\tan(G)G' + \frac{H}{s}\right) = 0 \\ &sH\cos(G)F' = 0 \\ &s\sin(G)H' + sH\cos(G)G' + H\sin(G) = 0 \\ &s\cos^2(G)HH' - s\sin(G)\cos(G)H^2G' + 3sAA' - H^2\sin^2(G) = 0. \end{aligned}$$

We have calculated the following solutions. If $H\cos(G) \neq 0$, then H is implicitly defined by

$$(-3C_4)^{1/2}(s^2H^6 - C_2^2)^{1/6} sH + C_3 - s = 0$$

and

$$\begin{aligned} &G = \cos^{-1}(s^{-1}H^{-1}(s^2H^2 - C_2^2)^{1/2}) \qquad F = C_1 \\ &A = C_4(s^2H^2 - C_2^2)^{-1/6} \end{aligned}$$

where $C_4 < 0$. Since we have an explicit expression for H, we substitute F, G, H and A into equations (4.20) in order to get an invariant solution of (1.4). If $H\cos(G) = 0$, two solutions are possible:

(i)

$$u = \frac{x - F(s)}{t} \qquad v = \frac{y}{t} \qquad w = \frac{z}{t} \qquad \text{and} \qquad a = \frac{C_4}{t}$$

(ii)

$$u = \frac{u - F(s)}{t} \qquad v = \frac{y}{t} + \epsilon\frac{C_3 z}{t^2 s^2} \qquad w = \frac{z}{t} - \epsilon\frac{C_3 y}{t^2 s^2}$$
$$\text{and} \qquad a = \epsilon\frac{(3C_4 s^2 - C_3^2)}{3^{1/2} st} \ (\epsilon = \pm 1)$$

where F is an arbitrary function of s. These solutions represent centred waves with conical symmetry. Some of them have been studied by Burnat [11, 12] and Rozdestvenski and Janenko [13].

Example 6. Algebraic solutions. We consider the algebra

$$\{K_1 + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3,\ \ K_2 - \alpha_2 P_1 + \beta_1 P_2 + \beta_2 P_3,\ \ K_3 - \alpha_3 P_1 - \beta_2 P_2\}$$

where $\alpha_2 > 0$, $\beta_2 > 0$; $\alpha_1, \alpha_3, \beta_1 \in \mathbb{R}$. The invariants are the symmetry variable t,

$$F(t) = x - (t + \alpha_1)u + \alpha_2 v + \alpha_3 w$$

$$G(t) = y - \alpha_2 u - (t + \beta_1)v + \beta_2 w$$

$$H(t) = z - \alpha_3 u - \beta_2 v - tw$$

and $A(t) = a$. From these invariants, we obtain the following expressions for the dependent variables:

$$u = \frac{(F - x)(t(t + \beta_1) + \beta_2^2) + (G - y)(\alpha_2 t - \alpha_3\beta_2) + (H - z)(\alpha_3(t + \beta_1) + \alpha_2\beta_2)}{\Delta_1}$$

$$v = \frac{(x - F)(\alpha_2 t + \alpha_3\beta_2) + (G - y)(t(t + \alpha_1) + \alpha_3^2) + (H - z)(\beta_2(t + \alpha_1) - \alpha_2\alpha_3)}{\Delta_1}$$

$$\begin{aligned} w = {} & [(x - F)(-\alpha_3(t + \beta_1) + \alpha_2\beta_2) - (G - y)(\beta_3(t + \alpha_1) + \alpha_3\alpha_2) \\ & +(H - z)((t + \alpha_1)(t + \beta_1) + \alpha_2^2)](\Delta_1)^{-1} \end{aligned}$$

where

$$\Delta_1 = t^3 + (\beta_1 + \alpha_1)t^2 + (\alpha_1\beta_1 + \beta_2^2 + \alpha_3^2 + \alpha_2^2)t + \alpha_3^2\beta_1 + \alpha_1\beta_2^2.$$

The reduced system is

$$A' + \frac{A}{k}\frac{(3t^2 + 2t(\beta_1 + \alpha_1) + \alpha_1\beta_1 + \beta_2^2 + \alpha_3^2 + \alpha_2^2)}{(t^3 + (\beta_1 + \alpha_1)t^2 + (\alpha_1\beta_1 + \beta_2^2 + \alpha_3^2 + \alpha_2^2)t + \alpha_3^2\beta_1 + \alpha_1\beta_2^2)} = 0$$

$$(t(t + \beta_1) + \beta_2^2)F' + (\alpha_2 t - \alpha_3\beta_2)G' + (\alpha_3(t + \beta_1) + \alpha_2\beta_2)H' = 0$$

$$-(\alpha_2 t + \alpha_3\beta_2)F' + (t(t + \alpha_1) + \alpha_3^2)G' + (\beta_2(t + \alpha_1) - \alpha_2\alpha_3)H' = 0$$

$$(-\alpha_3(t + \beta_1) + \alpha_2\beta_2)F' - (\beta_3(t + \alpha_1) + \alpha_3\alpha_2)G' + ((t + \alpha_1)(t + \beta_1) + \alpha_2^2)H' = 0.$$

The solution has the form

$$u = \frac{(C_1 - x)(t(t + \beta_1) + \beta_2^2) + (C_2 - y)(\alpha_2 t - \alpha_3\beta_2) + (C_3 - z)(\alpha_3(t + \beta_1)\alpha_2\beta_2)}{\Delta_1}$$

$$v = \frac{(x - C_1)(\alpha_2 t + \alpha_3\beta_2) + (C_2 - y)(t(t + \alpha_1) + \alpha_3^2) + (C_3 - z)(\beta_2(t + \alpha_1) - \alpha_2\alpha_3)}{\Delta_1}$$

$$\begin{aligned} w = {} & [(x - C_1)(-\alpha_3(t + \beta_1) + \alpha_2\beta_2) - (C_2 - y)(\beta_3(t + \alpha_1) + \alpha_3\alpha_2) \\ & +(C_3 - z)((t + \alpha_1)(t + \beta_1) + \alpha_2^2)](\Delta_1)^{-1} \end{aligned}$$

$$a = C_4(-\Delta_1)^{-1/k}.$$

Then the solution is a rational function. In order that the function a be well defined, real and non-singular for every k, we must impose the condition $\Delta_1 < 0$. Then $t \in D$ where $D = (-\infty, a_1) \cup (a_2, a_3)$, when $\Delta_1(a_i) = 0$, $a_i \in \mathbb{R}$ $(i = 1, 2, 3)$, and $a_1 < a_2 < a_3$; $D = (-\infty, a_1)$, when a_1 is the only root of Δ_1. Therefore, the domain of definition of the solution is $D \times \mathbb{R}^3$. On this domain, no gradient catastrophe appears and no shock wave can be produced [14]

5. Final remarks

The main difference between the method of calculation of the PISs proposed by Ovsiannikov and our method lies in the choice of the groups from which we calculate the PISs. Ovsiannikov suggests that every symmetry group of a system of differential equations can generate PISs, for a given defect structure δ, even if the group fulfills the conditions which allow us to construct invariant solutions from this group. In this case, we should use the first method of calculation presented in section 2 in order to obtain these solutions (i.e. the method with the matrix of characteristics). But the solutions which are obtained are not genuine PISs. Indeed, we can always find coordinates in which these solutions are invariant.

To conclude, let us remark that the method developed here, to obtain PISs, can also be used to describe phenomena concerning the superposition of waves in nonlinear continuous media, and in some cases we are able to determine the points at which the gradient catastrophe takes place. An example is the group generated by $\{J_1, P_2, P_3\}$. We obtain the PIS

$$u = s \qquad v = \sqrt{G(r) - w^2} \qquad w = H(r) \qquad a = k^{-1}s + C_1 \tag{5.1}$$

where the Riemann invariants are

$$s = (C_1 t - x)(1 - \beta t)^{-1} \qquad \beta = k^{-1}(k+1)$$

and

$$r = (1 - \beta t)^{-1/\beta}(x + C_1(\beta - 1)^{-1}(t - 1))$$

where G and H are arbitrary functions of the invariant r, C_1 is an arbitrary constant and $\beta \neq -1$. The solution (5.1) has rank two and therefore represents non-scattering double Riemann waves [15].

Acknowledgments

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